Multivariable Calculus

Exercises: solutions

1. Calculate all partial derivative of the following functions:

   a.) $g(x,y) = (x-y)^2$

   $g_x = 2(x-y)$
   $g_y = -2(x-y)$ (the minus sign comes from using the chain rule).

   An alternative here is to first expand the square ('remove parentheses'), then compute g_x and g_y from the result. This is more work, however. (And definitely not advisable for ex.2.b.)

   b.) $f(x,y) = \sin(\ln(x)) + xy + y^2$

   $f_x = \cos(\ln x) \cdot \frac{1}{x} + y$ (use chain rule)
   $f_y = x + 2y$.

2. Calculate the gradient of the following functions:

   a.) $f(x_1,x_2) = \dfrac{\sqrt{x_1^2-1}}{x_2^4}$

   The desired gradient is $\begin{pmatrix} f_{x_1} \\ f_{x_2} \end{pmatrix}$. Write the square root part as $(x_1^2-1)^{1/2}$ to arrive at
   $f_{x_1} = \frac{1}{x_2^4}\frac{1}{2}(x_1^2-1)^{-1/2} \cdot 2x_1 = \frac{x_1}{x_2^4\sqrt{x_1^2-1}}$.
   $f_{x_2} = \sqrt{x_1^2-1} \cdot (-4)x_2^{-5} = -\frac{4\sqrt{x_1^2-1}}{x_2^5}$.

   b.) $g(x,y,z) = (xyz - x^2 + y^2 - z^2)^2$

   The desired gradient is $\begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix}$.
   $g_x = 2(xyz - x^2 + y^2 - z^2)(yz - 2x)$
   $g_y = 2(xyz - x^2 + y^2 - z^2)(xz + 2y)$
   $g_z = 2(xyz - x^2 + y^2 - z^2)(xy - 2z)$.

3. Given the following functions f and directions $\mathbf{u}$. Calculate the directional derivative at the given coordinates.

   a.) $f(x,y) = x^2 + \sqrt{xy}$ and $\mathbf{u} = (\frac{1}{2}, \frac{1}{2})$ at position $(\sqrt{2}, 2\sqrt{2})$
   $f_x = 2x + \frac{1}{2}(xy)^{-1/2} \cdot y = 2x + \frac{y}{2\sqrt{xy}} = 2x + \frac{1}{2}\sqrt{\frac{y}{x}}$
   $f_y = \frac{1}{2}(xy)^{-1/2} \cdot x = \frac{1}{2}\sqrt{\frac{x}{y}}$.
   The general directional derivative is
   $\frac{1}{||u||}(u \cdot \nabla f) = \sqrt{2}(\frac{1}{4}\sqrt{\frac{x}{y}} + x + \frac{1}{4}\sqrt{\frac{y}{x}})$
   Filling in $(x,y) = (\sqrt{2}, 2\sqrt{2})$, we get
   $\frac{1}{||u||}(u \cdot \nabla f) = \sqrt{2}(\frac{1}{4}\sqrt{\frac{1}{2}} + \sqrt{2} + \frac{1}{4}\sqrt{2}) = \sqrt{2}(\frac{1}{8}\sqrt{2} + \sqrt{2} + \frac{1}{4}\sqrt{2}) = \frac{11}{8}\sqrt{2}\sqrt{2} = \frac{22}{8} = 2\frac{3}{4}$.

   b.) $f(x,y,z) = x + y + z$ and $\mathbf{u} = (2,0,1)$ at position (π, π^2, π^3)

$\nabla f = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\frac{1}{||u||}(u \cdot \nabla f) = \frac{1}{\sqrt{5}}(2 \cdot 1 + 0 \cdot 1 + 1 \cdot 1) = \frac{3}{\sqrt{5}} = \frac{3}{5}\sqrt{5}$.
Note that the directional derivative always has this value, regardless of the x,y, and z coordinates used.

4. Calculate the following multiple integrals:

   a.) $\int_1^2 \int_1^2 (x^2 + xy^2)\, dydx$

   $$\int_1^2 \int_1^2 (x^2 + xy^2)\, dydx = \int_1^2 \left[yx^2 + \frac{1}{3}xy^3\right]_1^2 dx = \int_1^2 ((2x^2 + \frac{8}{3}x) - (x^2 + \frac{1}{3}x))dx =$$
   $$\int_1^2 (x^2 + \frac{7}{3}x)dx = \left[\frac{1}{3}x^3 + \frac{7}{6}x^2\right]_1^2 = \frac{16}{6} + \frac{28}{6} - \frac{2}{6} - \frac{7}{6} = \frac{35}{6} = 5\frac{5}{6}.$$

   b.) $\int_0^\pi \int_{-\pi/2}^{\pi/2} \int_2^4 \left(z^3 \sin(y) \cos(x)\right) dzdydx$

   $\int_0^\pi \int_{-\pi/2}^{\pi/2} \int_2^4 \left(z^3 \sin(y) \cos(x)\right) dzdydx =$
   $\int_0^\pi \int_{-\pi/2}^{\pi/2} \sin y \cos x \left[\frac{1}{4}z^4\right]_2^4 dydx =$
   $\int_0^\pi \int_{-\pi/2}^{\pi/2} \sin y \cos x \frac{1}{4}(256 - 16)dydx =$
   $60 \int_0^\pi \int_{-\pi/2}^{\pi/2} \sin y \cos x dydx =$
   $-60 \int_0^\pi \cos x(-) \left[\cos y\right]_{-\pi/2}^{\pi/2} dx =$
   $-60 \int_0^\pi \cos x(0 - 0)dx = 0.$

5. Find the stationary points of the following functions:

   a.) $f(x, y) = \sin(x)\sin(y)$ on the domain $[-\pi, \pi] \times [-\pi, \pi]$

   $\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} \sin y \cos x \\ \sin x \cos y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ if
   $(\sin y \cos x = 0) \wedge (\sin x \cos y = 0)$, in the given domain this means
   $(x = \pm\frac{\pi}{2} \vee y = 0 \vee y = \pm\pi) \wedge (x = \pm\pi \vee x = 0 \vee y = \pm\frac{\pi}{2})$.
   Fishing out all valid coordinate pairs gives us 13 stationary points:

$(-\frac{\pi}{2}, -\frac{\pi}{2})$	$(-\pi, -\pi)$
$(-\frac{\pi}{2}, +\frac{\pi}{2})$	$(-\pi, 0)$
$(+\frac{\pi}{2}, -\frac{\pi}{2})$	$(-\pi, +\pi)$
$(+\frac{\pi}{2}, +\frac{\pi}{2})$	$(0, -\pi)$
	$(0, 0)$
	$(0, +\pi)$
	$(+\pi, -\pi)$
	$(+\pi, 0)$
	$(+\pi, +\pi)$

   b.) $f(x, y) = x^2y - xy^2 + 5$

   $\nabla f = \begin{pmatrix} 2xy - y^2 \\ x^2 - 2xy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ if
   $(2xy - y^2 = 0) \wedge (x^2 - 2xy = 0)$
   $(y(2x - y) = 0) \wedge (x(x - 2y) = 0)$
   $(y = 0 \vee x = \frac{1}{2}y) \wedge (x = 0 \vee x = 2y)$, the only possible point here is $(0, 0)$.

   c.) $f(x, y, z) = x^2 + y^2 + z^2 - 4x + 8y - 6z + 29$

   $\nabla f = \begin{pmatrix} 2x - 4 \\ 2y + 8 \\ 2z - 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ if $(x, y, z) = (2, -4, 3)$.